

# INFINITE PRODUCTS OF NONNEGATIVE $2 \times 2$ MATRICES BY NONNEGATIVE VECTORS

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**ABSTRACT.** Given a finite set  $\{M_0, \dots, M_{d-1}\}$  of nonnegative  $2 \times 2$  matrices and a non-negative column-vector  $V$ , we associate to each  $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$  the sequence of the column-vectors  $\frac{M_{\omega_1} \dots M_{\omega_n} V}{\|M_{\omega_1} \dots M_{\omega_n} V\|}$ . We give the necessary and sufficient condition on the matrices  $M_k$  and the vector  $V$  for this sequence to converge for all  $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$  such that  $\forall n, M_{\omega_1} \dots M_{\omega_n} V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

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## INTRODUCTION

Let  $\mathcal{M} = \{M_0, \dots, M_{d-1}\}$  be a finite set of nonnegative  $2 \times 2$  matrices and  $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  a non-negative column-vector. We use the notation  $Y_n = Y_n^\omega := M_{\omega_1} \dots M_{\omega_n}$  and give the necessary and sufficient condition for the pointwise convergence of  $\frac{Y_n V}{\|Y_n V\|}$ ,  $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$

such that  $Y_n V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  for any  $n$ , where  $\|\cdot\|$  is the norm-sum. The idea of the proof is that, if the conditions are satisfied, either both columns of  $Y_n$  tends to the same limit, or they tend to different limits with different orders of growth, so in case  $V$  is positive the limit points of  $\frac{Y_n V}{\|Y_n V\|}$  only depend on the limit of the dominant column. This problem is

obviously very different from the one of the convergence of  $\frac{Y_n}{\|Y_n\|}$ , or the convergence of the  $Y_n$  itselfs, see the introduction of [5] for some counterexamples and [8, Proposition 1.2] for the infinite products of  $2 \times 2$  stochastic matrices.

The conditions for the pointwise convergence of  $\frac{Y_n V}{\|Y_n V\|}$  also differ from the conditions for its uniform convergence, see [2]. The uniform convergence can be used for the multifractal analysis of some continuous singular measures called Bernoulli convolutions (see [6] for the

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Bernoulli convolutions and [1] for their multifractal analysis). We study such measures in [2], [3] and [4]. The Birkhoff's contraction coefficient [7, Chapter 3] that we use in [3] and [5] but not here, is really not of great help to solve the main difficulties. Moreover the theorem that gives the value of this coefficient is difficult to prove (see [7, §3.4]) even in the case of  $2 \times 2$  matrices. In [2] we use some other contraction coefficient quite more easy to compute ([2, Proposition 1.3]).

# 1. CONDITION FOR THE POINTWISE CONVERGENCE OF $\frac{Y_n V}{\|Y_n V\|}$

**Proposition 1.1.** *The sequence  $\frac{Y_n V}{\|Y_n V\|}$  converges for any  $\omega \in \{0, \dots, d-1\}^{\mathbb{N}}$  such that*

*$\forall n, Y_n V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , if and only if at least one of the following conditions holds:*

*(i)  $V$  has positive entries and it is an eigenvector of any invertible matrix of the form  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  or  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$  that belongs to  $\mathcal{M}$ .*

*(ii) Any invertible matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$  satisfies  $a > 0$  and, if  $b = c = 0$ ,  $a \geq d$ .*

*(iii) Any invertible matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$  satisfies  $d > 0$  and, if  $b = c = 0$ ,  $d \geq a$ .*

*(iv)  $V$  has a null entry and all the invertible matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$  satisfy  $ad > 0$ .*

*Proof.* Let  $\omega \in \{0, \dots, d-1\}^{\mathbb{N}}$ . If there exists  $N$  such that  $\det M_{\omega_N} = 0$ , the column-vectors  $Y_N V, Y_{N+1} V, \dots$  are collinear and  $\frac{Y_n V}{\|Y_n V\|}$  is constant for  $n \geq N$ . So we look only at the  $\omega \in \{0, \dots, d-1\}^{\mathbb{N}}$  such that  $\forall n, \det M_{\omega_n} \neq 0$ . In order to use only matrices with positive determinant we set  $\Delta := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and

$$A_n = A_n^\omega := \begin{cases} M_{\omega_n} & \text{if } \det Y_{n-1} > 0 \text{ (or } n = 1) \text{ and } \det M_{\omega_n} > 0 \\ M_{\omega_n} \Delta & \text{if } \det Y_{n-1} > 0 \text{ (or } n = 1) \text{ and } \det M_{\omega_n} < 0 \\ \Delta M_{\omega_n} & \text{if } \det Y_{n-1} < 0 \text{ and } \det M_{\omega_n} < 0 \\ \Delta M_{\omega_n} \Delta & \text{if } \det Y_{n-1} < 0 \text{ and } \det M_{\omega_n} > 0. \end{cases}$$

We set also

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} := A_n \quad \text{and} \quad \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} := A_1 \dots A_n = \begin{cases} Y_n & \text{if } \det Y_n > 0 \\ Y_n \Delta & \text{if } \det Y_n < 0. \end{cases}$$

The matrices  $A_n$  belong to the set

$$\mathcal{M}^+ := \{M ; \exists i, j, k, M = \Delta^i M_k \Delta^j \text{ and } \det M > 0\}.$$

Since  $\det A_n > 0$  we have  $a_n d_n p_n s_n \neq 0$ . If  $\{n; A_n \text{ not diagonal}\}$  is infinite we index this set by an increasing sequence  $n_1 < n_2 < \dots$ . We have  $b_{n_1} \neq 0$  or  $c_{n_1} \neq 0$ ; both cases are equivalent because, using the set of matrices  $\mathcal{M}' = \Delta \mathcal{M} \Delta$  and defining similarly  $Y'_n$  and  $A'_n = \begin{pmatrix} a'_n & b'_n \\ c'_n & d'_n \end{pmatrix}$  from this set, we have  $Y'_n = \Delta Y_n \Delta$ ,  $A'_n = \Delta A_n \Delta$  and  $b'_{n_1} = c_{n_1}$ . So we can suppose  $b_{n_1} \neq 0$ ; we deduce  $q_n \neq 0$  by induction on  $n \geq n_1$ . The sequences defined for any  $n \geq n_1$  by

$$u_n = \frac{r_n}{p_n}, \quad v_n = \frac{s_n}{q_n}, \quad w_n = \frac{q_n}{p_n}, \quad x_n = \begin{cases} v_2/v_1 & \text{if } \det Y_n > 0 \\ v_1/v_2 & \text{if not} \end{cases} \quad \text{and} \quad \lambda_n = (1 + w_n x_n)^{-1}$$

satisfy  $0 \leq u_n < v_n < \infty$ ,  $0 < w_n < \infty$  and

if the entries of  $V$  are positive,  $0 < x_n < \infty$  and  $0 < \lambda_n < 1$   
if not,  $x_n \in \{0, \infty\}$  and  $\lambda_n \in \{0, 1\}$  according to the sign of  $\det Y_n$ .

Since we have assumed that  $Y_n V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , the ratio  $\frac{(Y_n V)_2}{(Y_n V)_1}$  exists in  $[0, \infty]$  and we have to prove that it has a finite or infinite limit when  $n \rightarrow \infty$ . If  $A_n$  is not eventually diagonal we have for  $n \geq n_1$

$$(1) \quad \frac{(Y_n V)_2}{(Y_n V)_1} = \lambda_n u_n + (1 - \lambda_n) v_n \in I_n := [u_n, v_n] \text{ and } I_n \supseteq I_{n+1}.$$

An immediate consequence is the following lemma:

**Lemma 1.1.** *Suppose  $A_n$  is not eventually diagonal, then*

- (i) *the sequences  $(u_n)$  and  $(v_n)$  converge in  $\mathbb{R}$  and the sequence  $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$  is bounded;*
- (ii)  *$\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$  converges if  $\lim_{n \rightarrow \infty} |I_n| = 0$ ;*
- (iii) *if  $V$  has positive entries,  $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$  converges if  $w_n$  has limit 0 or  $\infty$ ;*
- (iv) *if  $V$  has a null entry, the necessary and sufficient condition for the convergence of  $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$  is that  $\lim_{n \rightarrow \infty} |I_n| = 0$  or the sign of  $\det Y_n$  is eventually constant.*

We also define for  $n > n_1$

$$\alpha_n = \left(1 + \frac{c_n}{a_n} w_{n-1}\right)^{-1}, \quad \beta_n = \left(1 + \frac{b_n}{d_n} (w_{n-1})^{-1}\right)^{-1}, \quad \gamma_n = 1 - \frac{c_n}{a_n} \frac{b_n}{d_n}$$

that belong to  $]0, 1]$  and satisfy

$$(2) \quad \begin{aligned} |I_n| &= \alpha_n \beta_n \gamma_n |I_{n-1}| \\ w_n &= \frac{d_n}{a_n} \frac{\alpha_n}{\beta_n} w_{n-1} \end{aligned}$$

so  $\prod_{n>n_1} \alpha_n \beta_n \gamma_n = \lim_{n \rightarrow \infty} \frac{|I_n|}{|I_{n_1}|}$  is positive if and only if  $\lim_{n \rightarrow \infty} |I_n| > 0$ . Using the equivalents of  $\log \alpha_n$ ,  $\log \beta_n$  and  $\log \gamma_n$ ,

$$(3) \quad \lim_{n \rightarrow \infty} |I_n| > 0 \Leftrightarrow \sum \frac{c_n}{a_n} w_{n-1} < \infty, \quad \sum \frac{b_n}{d_n} (w_{n-1})^{-1} < \infty \quad \text{and} \quad \sum \frac{c_n}{a_n} \frac{b_n}{d_n} < \infty.$$

The set of indexes  $\{n; A_n \text{ not diagonal}\}$  is the union of

$$L^\omega = \{n; c_n \neq 0\} \quad \text{and} \quad U^\omega = \{n; b_n \neq 0\}.$$

Moreover, since  $A_n$  belongs to the finite set  $\mathcal{M}^+$  there exists  $K > 0$  such that

$$L^\omega = \left\{ n; \frac{1}{K} \leq \frac{c_n}{a_n} \leq K \right\} \quad \text{and} \quad U^\omega = \left\{ n; \frac{1}{K} \leq \frac{b_n}{d_n} \leq K \right\}.$$

We deduce a simpler formulation of (3):

$$(4) \quad \lim_{n \rightarrow \infty} |I_n| > 0 \Leftrightarrow \sum_{n \in L^\omega} w_{n-1} < \infty, \quad \sum_{n \in U^\omega} (w_{n-1})^{-1} < \infty \quad \text{and} \quad L^\omega \cap U^\omega \text{ is finite.}$$

In view of Lemma 1.1 we may suppose from now that  $\lim_{n \rightarrow \infty} |I_n| > 0$ . Since  $L^\omega \cap U^\omega = \{n; A_n \text{ positive}\}$  is finite, for  $n$  large enough the matrix  $A_n$  is lower triangular if  $n \in L^\omega$ , upper triangular if  $n \in U^\omega$ , diagonal if  $n \notin L^\omega \cup U^\omega$ . When  $A_n$  is diagonal the second relation in (2) becomes  $w_n = \frac{d_n}{a_n} w_{n-1}$ ; consequently any integer  $n$  in an interval  $]n_i, n_{i+1}[$  with  $i$  large enough satisfies

$$(5) \quad \frac{w_n}{w_{n_i}} = \prod_{n_i < j \leq n} \frac{d_j}{a_j}.$$

Moreover if  $L^\omega$  is infinite, (4) implies that  $w_{n-1}$  has limit to 0 when  $L^\omega \ni n \rightarrow \infty$ , and  $w_n$  also has limit 0 because  $w_n = \frac{d_n w_{n-1}}{a_n + c_n w_{n-1}}$  for any  $n \in L^\omega \setminus U^\omega$ . We have a similar property if  $U^\omega$  is infinite, so

$$(6) \quad \begin{array}{l} \text{if } L^\omega \text{ is infinite, } w_{n-1} \rightarrow 0 \text{ and } w_n \rightarrow 0 \text{ for } L^\omega \ni n \rightarrow \infty; \\ \text{if } U^\omega \text{ is infinite, } w_{n-1} \rightarrow \infty \text{ and } w_n \rightarrow \infty \text{ for } U^\omega \ni n \rightarrow \infty. \end{array}$$

**First case:** Suppose that (i) holds. Then the diagonal matrices of  $\mathcal{M}$  are collinear to the unit matrix. If at least one matrix of  $\mathcal{M}$  has the form  $M_k = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$  with  $bc \neq 0$ , its nonnegative eigenvalue – namely  $\left(\frac{\sqrt{b}}{\sqrt{c}}\right)$  – is collinear to  $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  hence there exists  $\lambda \in \mathbb{R}$  such that  $M_k = \lambda \begin{pmatrix} 0 & v_1^2 \\ v_2^2 & 0 \end{pmatrix}$ .

Notice that if  $A_n$  is diagonal from a rank  $N$ , the matrix  $M_{\omega_n}$  has the form  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  or  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$  hence it has  $V$  as eigenvector; consequently  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  converges because it is  $\frac{(Y_N V)_2}{(Y_N V)_1}$  for any  $n \geq N$ .

Suppose now  $A_n$  is non-diagonal for infinitely many  $n$ . We apply (5) on each interval  $]n_i, n_{i+1}[$  (if non empty), for  $i$  large enough. Among the integers  $n \in ]n_i, n_{i+1}[$  we consider the ones for which  $\det M_{\omega_n} < 0$ . For such  $n$  the matrix  $A_n$  is alternately  $M_{\omega_n} \Delta$  and  $\Delta M_{\omega_n}$ , hence alternately proportional to  $\begin{pmatrix} v_1^2 & 0 \\ 0 & v_2^2 \end{pmatrix}$  and to  $\begin{pmatrix} v_2^2 & 0 \\ 0 & v_1^2 \end{pmatrix}$  and, according to (5),

$$(7) \quad n_i \leq n < n_{i+1} \Rightarrow \frac{w_n}{w_{n_i}} \in \left\{ \frac{v_1^2}{v_2^2}, \frac{v_2^2}{v_1^2}, 1 \right\}.$$

In particular this relation holds for  $n = n_{i+1} - 1$ . One deduce – according to (6) – that there do not exist infinitely many  $i$  such that  $n_i \in L^\omega$  and  $n_{i+1} \in U^\omega$ . Thus  $n_i \in L^\omega$  for  $i$  large enough (resp.  $n_i \in U^\omega$  for  $i$  large enough) and, according to (6) and (7),  $\lim_{n \rightarrow \infty} w_n = 0$  (resp.  $\lim_{n \rightarrow \infty} w_n = \infty$ ). In view of Lemma 1.1(iii),  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  converges.

**Second case:** Suppose that (ii) holds (if (iii) holds the proof is similar).

Suppose first the  $M_{\omega_n}$  are diagonal from a rank  $N$ . From the hypothesis (ii) there exists  $\delta_n, \delta'_n$  such that  $M_{\omega_N} \dots M_{\omega_n} V = \begin{pmatrix} \delta_n v_1 \\ \delta'_n v_2 \end{pmatrix}$  and  $\delta_n \geq \delta'_n$ . Since the  $M_{\omega_i}$  belong to a finite set we have  $\lim_{n \rightarrow \infty} \frac{\delta_n}{\delta'_n} = \infty$ , or  $\frac{\delta_n}{\delta'_n}$  is eventually constant in case  $M_{\omega_n}$  is eventually the unit matrix, or  $\delta'_n = 0 \neq \delta_n$  for  $n$  large enough. Denoting by  $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$  the matrix  $M_{\omega_1} \dots M_{\omega_{N-1}}$ , we have  $\frac{(Y_n V)_2}{(Y_n V)_1} = \frac{r\delta_n v_1 + s\delta'_n v_2}{p\delta_n v_1 + q\delta'_n v_2}$  hence  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  converges in all the cases. Suppose now  $M_{\omega_n}$  is non-diagonal for infinitely many  $n$ . There exists from (6) an integer  $\kappa$  such that

$$(8) \quad i \geq \kappa \Rightarrow \begin{cases} w_{n_{i-1}} < 1 \text{ and } w_{n_i} < 1 & \text{if } n_i \in L^\omega \\ w_{n_{i-1}} > 1 \text{ and } w_{n_i} > 1 & \text{if } n_i \in U^\omega \end{cases}$$

and such that the  $A_n$  are diagonal for  $n \in ]n_i, n_{i+1}[$ ,  $i \geq \kappa$ . According to (ii), for such values of  $n$  the matrix  $M_{\omega_n}$  is diagonal and  $A_n = M_{\omega_n}$  with  $a_n \geq d_n$ , or  $A_n = \Delta M_{\omega_n} \Delta$  with  $a_n \leq d_n$ .

If there exists  $i \geq \kappa$  such that  $n_i \in L^\omega$  and  $n_{i+1} \in U^\omega$ ,  $\det Y_{n_i}$  is necessarily negative: otherwise  $A_n$  should be equal to  $M_{\omega_n}$  for  $n \in ]n_i, n_{i+1}[$ ,  $\frac{d_n}{a_n} \leq 1$  and, by (5),  $w_{n_{i+1}-1} \leq w_{n_i} < 1$  in contradiction with (8).

Now  $M_{\omega_{n_{i+1}}}$  has positive determinant, otherwise it should have the form  $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$  and  $A_{n_{i+1}} = \Delta M_{\omega_{n_{i+1}}} = \begin{pmatrix} c & 0 \\ a & b \end{pmatrix}$  in contradiction with  $n_{i+1} \in U^\omega$ .

We have again  $\frac{d_n}{a_n} \geq 1$  for  $n \in ]n_{i+1}, n_{i+2}[$  and consequently  $w_{n_{i+2}-1} \geq w_{n_{i+1}} > 1$ ; so, by induction,  $n_j \in U^\omega$  and  $\det Y_{n_j} < 0$  for any  $j \geq i+1$ . From (5)  $w_n$  lies between  $w_{n_j}$  and  $w_{n_{j+1}-1}$  for any  $n \in ]n_j, n_{j+1}[$  and  $j$  large enough, and from (6) its limit is infinite. Distinguishing the cases where  $V$  has positive entries or  $V$  has a null entry,  $\frac{(Y_n V)_2}{(Y_n V)_1}$  converges by Lemma 1.1.

The conclusion is the same if there do not exist  $i \geq \kappa$  such that  $n_i \in L^\omega$  and  $n_{i+1} \in U^\omega$ , because in this case  $n_i \in L^\omega$  for  $i$  large enough, or  $n_i \in U^\omega$  for any  $i \geq \kappa$ .

**Third case:** Suppose (iv) holds. As we have seen, from (4)  $A_n$  is eventually triangular or diagonal, and  $M_{\omega_n}$  also is because – by (iv) –  $\mathcal{M}$  do not contain invertible matrices of the form  $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$  or  $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ . We deduce that the sign of  $\det Y_n$  is eventually constant. If  $A_n$  is not eventually diagonal the sequence  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  converges by Lemma 1.1(iv) and, if  $A_n$  is, the sequence  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  is eventually constant.

**Fourth case:** Suppose that the set  $\mathcal{M}$  do not satisfy (i), (ii), (iii) nor (iv), and that at least one matrix of this set, let  $M_k$ , has the form  $M_k = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$  with  $bc \neq 0$ ; let us prove that  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  diverges.

– Suppose first there exists a matrix  $M_k$  of this form that do not have  $V$  as eigenvector; we chose as counterexample the constant sequence defined by  $\omega_n = k$  for any  $n$ :  $Y_{2n}$  is collinear to the unit matrix, hence  $Y_{2n}V$  is collinear to  $V$  and  $Y_{2n+1}V$  to  $M_k V$ , so  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  diverges.

– Suppose now that all the matrices of  $\mathcal{M}$  of the form  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$  with  $bc \neq 0$  have  $V$  as eigenvector that is,  $V$  is collinear to  $\begin{pmatrix} \sqrt{b} \\ \sqrt{c} \end{pmatrix}$  for all such matrix. Since (i) do not hold, at least one matrix  $M_h$  of  $\mathcal{M}$  is diagonal with nonnull and distinct diagonal entries. In

this case  $M_h M_k$  has the form  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$  but do not have  $V$  as eigenvector. We recover the previous case; more precisely the counterexample is defined by  $\omega_{2n-1} = h$  and  $\omega_{2n} = k$  for any  $n \in \mathbb{N}$ .

**Fifth case:** Suppose that  $\mathcal{M}$  do not satisfy (i), (ii), (iii) nor (iv), and that no matrix of the form  $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$  with  $bc \neq 0$  belongs to  $\mathcal{M}$ . Since (i) do not hold, at least one matrix of this set has the form  $M_k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  with  $ad \neq 0$  and  $a \neq d$ . We suppose that  $a > d$  and we use the negation of (ii) (in case  $a < d$  we use similarly the negation of (iii)). According to the negation of (ii) there exists in  $\mathcal{M}$  at least one matrix of the form  $M_h = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$  with  $\beta\gamma\delta \neq 0$ , or one of the form  $M_\ell = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$  with  $0 < \alpha < \delta$ .

– Consider first the case where  $\mathcal{M}$  contains some matrices  $M_k$  and  $M_h$  as above. Let  $(n_i)_{i \in \mathbb{N}}$  be an increasing sequence of positive integers with  $n_1 = 1$ , and  $\omega$  the sequence defined by  $\omega_n = h$  for  $n \in \{n_1, n_2, \dots\}$  and  $\omega_n = k$  otherwise.

For  $i$  odd,  $A_{n_i}$  is lower-triangular and  $\forall n \in ]n_i, n_{i+1}[$ ,  $A_n = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$ ,  $a_n = d$  and  $d_n = a$ .

For  $i$  even,  $A_{n_i}$  is upper-triangular and  $\forall n \in ]n_i, n_{i+1}[$ ,  $A_n = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ ,  $a_n = a$  and  $d_n = d$ .

Using (5) for  $n = n_{i+1} - 1$  and choosing  $n_{i+1} - n_i$  large enough one has  $w_{n_{i+1}-1} \geq 2^i$  if  $i$  is odd,  $w_{n_{i+1}-1} \leq 2^{-i}$  if  $i$  is even, so the three conditions in (4) are satisfied and the interval  $\cap I_n$  is not reduced to one point. If the entries of  $V$  are positive, the first relation in (1) and the definition of  $\lambda_n$  imply that  $\liminf_{n \rightarrow \infty} \left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  is the lower bound of this interval

and  $\limsup_{n \rightarrow \infty} \left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  its upper bound, so the sequence  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  diverges. If  $V$  has a

null entry, the divergence of  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  results from Lemma 1.1(iv).

– In case  $\mathcal{M}$  contains some matrices  $M_k$  and  $M_\ell$  as above, one defines  $\omega$  from a sequence  $i_1 = 1 < i_2 < i_3 < \dots$  by setting, for  $j \geq 1$  and  $i_j \leq n < i_{j+1}$ ,

$$\omega_n = \begin{cases} k & \text{if } j \text{ even} \\ \ell & \text{if } j \text{ odd.} \end{cases}$$

The diagonal matrix  $Y_n$  can be easily computed, and  $\left( \frac{(Y_n V)_2}{(Y_n V)_1} \right)$  obviously diverges if one choose the  $i_{j+1} - i_j$  large enough.

If  $V$  has a null entry, since (iv) do not hold  $\mathcal{M}$  contains at least one matrix of the form  $M_h = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ ,  $\beta\gamma\delta \neq 0$  or  $M_{h'} = \begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}$ ,  $\alpha\beta\gamma \neq 0$ , or  $M_{h''} = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ ,  $\beta\gamma \neq 0$ . We already know that  $\left(\frac{(Y_n V)_2}{(Y_n V)_1}\right)$  diverges if  $\mathcal{M}$  contains  $M_k$  and  $M_h$ . Similarly it diverges if  $\mathcal{M}$  contains  $M_\ell$  and  $M_{h'}$ . If  $\mathcal{M}$  contains  $M_k$  and  $M_{h''}$  the counterexample is given – from a sequence  $i_1 = 1 < i_2 < i_3 < \dots$  – by  $\omega_{i_j} = h''$  and  $\omega_n = k$  for  $n \in ]i_j, i_{j+1}[$ ,  $j \in \mathbb{N}$ :  $\frac{(Y_n V)_2}{(Y_n V)_1}$  is alternately 0 and  $\infty$  because  $Y_{i_j}$  has the form  $\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$  for  $j$  odd and  $\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}$  for  $j$  even.  $\square$

## REFERENCES

- [1] D-J. FENG & E. OLIVIER, Multifractal analysis of weak Gibbs measures and phase transition – application to some Bernoulli convolutions, *Ergodic Theory Dyn. Syst.* **23** (2003), 1751–1784, [download](#).
- [2] E. OLIVIER, A. THOMAS, Infinite products of  $2 \times 2$  matrices and the Gibbs properties of Bernoulli convolutions (2006), [download](#).
- [3] E. OLIVIER, A. THOMAS, Infinite Convolution of Bernoulli Measures, PV numbers and related problems in the dynamics of Fractal Geometry, *Compte rendu d'un exposé à l'École Plurithématique de Théorie Ergodique II* (2006), [download](#).
- [4] E. OLIVIER, A. THOMAS, How to prove that some Bernoulli convolution is weak Gibbs (2010), [download](#).
- [5] E. OLIVIER, A. THOMAS, Asymptotic properties of the columns in the products of nonnegative matrices (2009), [download](#).
- [6] Y. PERES, W. SCHLAG & B. SOLOMYAK, Sixty years of Bernoulli convolutions, *Progress in Probability*, Birkhäuser Verlag **Vol. 46** (2000), 39–65, [download](#).
- [7] E. SENETA, Non-negative matrices and Markov chains, *Springer Series in Statistics. New York - Heidelberg - Berlin: Springer-Verlag* **XV** (1981), [partial download](#).
- [8] A. THOMAS, Can an infinite left-product of nonnegative matrices be expressed in terms of infinite left-products of stochastic ones? (2010), [download](#).

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